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Exact solution and Berry phase for the dissipative two-mode optical system

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Abstract
We have explored the dynamical algebraic structure of the dissipative two-mode system. On the basis of the algebraic structure, we have obtained the exact eigensolutions of the Liouvillian superoperator of its master equation by using an algebraic dynamical method. Then we investigate the Berry phase of the system within the density-matrix formalism.

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1. Introduction

Since the geometrical phase generated along a closed curve in a parameter space of a quantum system under the adiabatic evolution was explored by Berry [1], much attention has been paid to its extension and application. Aharonov and Anandan [2] have discussed the geometrical phase for any periodic quantum system beyond the adiabatic limit. Subsequently, Samuel and Bhandari [3] have transplanted the theorem from classical optical interference into quantum mechanics and obtained the most general geometrical phase called Pancharatnam phase [4]. The general formalism and properties of the geometrical phase based on the quantum kinematic theorem have been systematically developed by Mukunda *et al* [5]. The geometrical phase has been studied in the semiclassical context before, i.e. the atomic part is treated by the quantum mechanical theorem while the field driving the atom is treated classically. Recently Lawande *et al* have investigated the Pancharatnam phase for the system of a two-level atom interacting with a quantized field, and found that the Pancharatnam phase is related to the statistics of the field and atomic coherence [6]. On the other hand, the mass of the studies of the geometrical phase has been confined to the Hermitian quantum system [7]. Some other authors, on the basis of non-Hermitian Hamiltonians, have discussed the complex geometrical phase associated with dissipative systems [8–11]. Ellinas *et al* have investigated the geometrical phase in the model of optical resonance by using the density-matrix formalism [12].

The two-mode optical system under discussion is based on the coupled harmonic oscillators [13–15]. The model can describe the rotating-wave coupling part of the interaction between two modes of electromagnetic fields in the presence of strong classical pumping in a nonlinear crystal [16]. It is of great interest in quantum optics since it can be used as a quantum frequency converter. The two-mode system, moreover, can also be used to describe two coupled quantum cavities [17–19]. This system displays many interesting physical phenomena. For example, not only can it cause the energy exchange between the two modes, but can also change the coherence and statistical properties of the two modes [20]. Yet it can also be used to display the interaction-free measurement, the reversible decoherence of Schrödinger cat states and the modelling quantum switches [17–19, 21]. The quantum information transfer between the quantum cavities can also be studied by employing this system [16, 22]. Besides, the system has many nonclassical properties such as squeezing, photon statistics and entanglement [23–25]. It is of practical importance to study the above-mentioned properties more realistically under the effect of the dissipative background and the external driving fields. For this reason, we have generalized the two-mode system to the case where the dissipative background and two driving external fields are included together with the system.

The two-mode system has been used to investigate the physical realization of the vacuum induced geometrical phase in [26]. In this paper we use an algebraic dynamical method [27–31] to obtain the exact eigensolutions of the Liouvillian superoperator in its master equation of the generalized two-mode system and to discuss the Berry phase and its dependence on the external classical driving field and the dissipative background field. The geometrical phase of the system is very important for its application in quantum computation and other fields. The Liouvillian superoperator contains two parts: one from the Hermitian Hamiltonian of the system and the other related to the damping caused by the environment. In [12], since the eigensolutions of the total Liouvillian superoperator in their model of optical resonance are difficult to obtain, the authors just investigated the geometrical phase in the weak damping limit, taking the damping part as a perturbation. In this paper, beyond the weak damping limit, we discuss the Berry phase for the system on the basis of exact eigensolutions of the total Liouvillian superoperator with any damping strength. Our paper is organized as follows: in section 2 we describe the Hamiltonian and master equation for the dissipative two-mode optical system. In section 3 we explore the dynamical algebraic structure of the master equation of the system. The adiabatic solution to the master equation is obtained by diagonalizing the instantaneous Liouvillian superoperator in section 4. The geometrical phase for the dissipative two-mode optical system is discussed in section 5, and the conclusion is presented in section 6.

2. Hamiltonian and master equation for dissipative two-mode optical system

The Hamiltonian for the dissipative two-mode optical system driven by two external fields in the rotating wave approximation reads

$$\hat{H} = \hat{H}_0 + \hat{H}_S + \hat{H}_D + \hat{H}_R \tag{1}$$

where

$$\hat{H}_0 = \sum_{i=1}^{2} w_i \hat{a}_i^+ \hat{a}_i + \sum_{v} w_v \hat{b}_v^+ \hat{b}_v,$$

$$\hat{H}_S = F(t)\hat{a}_1^+\hat{a}_2 + F(t)^*\hat{a}_2^+\hat{a}_1,$$
$$\hat{H}_D = \sum_{j=1}^{2}\big(f_j(t)\hat{a}_j^+ + f_j(t)^*\hat{a}_j\big),$$
$$\hat{H}_R = \sum_v\big(g_{1v}\hat{b}_v^+\hat{a}_1 + \text{h.c.}\big) + \sum_v\big(g_{2v}\hat{b}_v^+\hat{a}_2 + \text{h.c.}\big).$$

In $\hat{H}_0$, the first term is the energy of the two-mode system and the second term is that for the reservoir. $\hat{H}_S$ describes the interaction between the two modes and $\hat{H}_D$ denotes the optical pumping on the two modes with external classical fields. $\hat{H}_R$ describes the coupling between the system and the reservoir. $\hat{a}_i$ $\big(\hat{a}_i^+\big)$ and $\hat{b}_v$ $\big(\hat{b}_v^+\big)$ are the destruction (creation) operators of the system and the reservoir with frequencies w_i and w_v, respectively. g_{iv} (g_{iv}^*) are the coupling constants between the system and the reservoir. $F(t)$ $(F(t)^*)$ and $f_j(t)$ $(f_j(t)^*)$ are the coupling constants between the two modes and amplitudes of the coherent driving fields, respectively. The Hamiltonian $\hat{H}_R$ in equation (1) can be rewritten as

$$\hat{H}_R = \sum_v\big(g_v\hat{b}_v^+\hat{C} + \text{h.c.}\big) \tag{2}$$

where $g_v = g_{2v}\sqrt{1+k^2}$ and $\hat{C} = \frac{1}{\sqrt{1+k^2}}(k\hat{a}_1 + \hat{a}_2)$. $k = \frac{g_{1v}}{g_{2v}}$ is supposed to be independent of the mode of the reservoir and is a real constant. Following the procedures in [32, 33] and in the Markov approximation, the master equation of the reduced density including $\hat{H}_R$ and discarding $\hat{H}_S$ and $\hat{H}_D$ is

$$\frac{\mathrm{d}\hat{\rho}_R}{\mathrm{d}t} = \gamma(2\hat{C}\hat{\rho}_R\hat{C}^+ - \hat{C}^+\hat{C}\hat{\rho}_R - \hat{\rho}_R\hat{C}^+\hat{C}), \tag{3}$$

where γ is the damping constant corresponding to the collective operators $\hat{C}$ (or $\hat{C}^+$) and is related to the coupling constant g_v. On including $\hat{H}_S$ and $\hat{H}_D$, the master equation of the reduced density for the total system driven by the Hamiltonian in equation (1) in the interaction picture is as follows in terms of $\hat{a}_i^+, \hat{a}_i$ $(i = 1, 2)$

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\mathrm{i}F(t)\big(\hat{a}_1^+\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_1^+\hat{a}_2\big) - \mathrm{i}F(t)\big(\hat{a}_2^+\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_2^+\hat{a}_1\big)\\
&- \sum_{j=1}^{2}\big[\mathrm{i}f_j(t)\big(\hat{a}_j^+\hat{\rho} - \hat{\rho}\hat{a}_j^+\big) + \mathrm{i}f_j^*(t)(\hat{a}_j\hat{\rho} - \hat{\rho}\hat{a}_j)\big]\\
&+ \sum_{i=1}^{2}\gamma_i\big(2\hat{a}_i\hat{\rho}\hat{a}_i^+ - \hat{a}_i^+\hat{a}_i\hat{\rho} - \hat{\rho}\hat{a}_i^+\hat{a}_i\big)\\
&+ \gamma_3\sum_{i,j=1(i\neq j)}^{2}\big(2\hat{a}_i\hat{\rho}\hat{a}_j^+ - \hat{a}_i^+\hat{a}_j\hat{\rho} - \hat{\rho}\hat{a}_i^+\hat{a}_j\big),\end{aligned} \tag{4}$$

where we assume that $F(t)$ is a real parameter. $\gamma_1 = \frac{\gamma k^2}{\sqrt{1+k^2}}$, $\gamma_2 = \frac{\gamma}{\sqrt{1+k^2}}$ and $\gamma_3 = \sqrt{\gamma_1\gamma_2}$ are the damping constants, respectively. Equation (4) can be written in a simple form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \sum_{i=1}^{2}\gamma_i\big(2\hat{A}_i\hat{\rho}\hat{A}_i^+ - \hat{A}_i^+\hat{A}_i\hat{\rho} - \hat{\rho}\hat{A}_i^+\hat{A}_i\big) + \sum_{i,j=1(i\neq j)}^{2}\big(2\gamma_3\hat{A}_i\hat{\rho}\hat{A}_j^+ - g\hat{A}_i^+\hat{A}_j\hat{\rho} - g^*\hat{\rho}\hat{A}_i^+\hat{A}_j\big). \tag{5}$$

The ‘displaced’ bosonic operators are defined as $\hat{A}_1 = \hat{a}_1 + x(t)$ $\big(\hat{A}_1^+ = \hat{a}_1^+ + x(t)^*\big)$ and $\hat{A}_2 = \hat{a}_2 + y(t)$ $\big(\hat{A}_2^+ = \hat{a}_2^+ + y(t)^*\big)$ satisfying the same $hw(4)$ commutation relation as $\big\{a_i, a_i^+\big\}$, where $x(t) = \frac{\mathrm{i}(gf_2-\gamma_2 f_1)}{g^2-\gamma_1\gamma_2}$, $y(t) = \frac{\mathrm{i}(gf_1-\gamma_1 f_2)}{g^2-\gamma_1\gamma_2}$ and $g(t) = \gamma_3 + \mathrm{i}F(t)$.

3. Dynamical algebraic structure of master equation

We introduce the left and right algebras based on the location where the destruction (creation) operators of the system stand to the left and right sides of the density operator. The left and right algebras of the algebra $hw(4)$ denoted by $hw(4)_l$ and $hw(4)_r$ for the two modes are given by [28–31]

$$hw(4)_l: \quad \left[\hat{n}_{il}, \hat{a}_{il}^{+}\right] = \hat{a}_{il}^{+}, \qquad [\hat{n}_{il}, \hat{a}_{il}] = -\hat{a}_{il}, \qquad \left[\hat{a}_{il}, \hat{a}_{il}^{+}\right] = 1, \tag{6}$$

$$hw(4)_r: \quad \left[\hat{n}_{ir}, \hat{a}_{ir}^{+}\right] = -\hat{a}_{ir}^{+}, \qquad [\hat{n}_{ir}, \hat{a}_{ir}] = \hat{a}_{ir}, \qquad \left[\hat{a}_{ir}, \hat{a}_{ir}^{+}\right] = -1, \quad i = 1 \text{ or } 2 \tag{7}$$

where the operators in the $hw(4)_l$ act from the left on the ket state $|\ \rangle$ and the operators in the $hw(4)_r$ from the right on the bra state $\langle\ |$. The $hw(4)_l$ ($hw(4)_r$) are isomorphic (anti-isomorphic) to $hw(4)$. Since the operators in the $hw(4)_l$ and $hw(4)_r$ algebras act in different spaces, they commute with each other as follows

$$[\hat{C}_l, \hat{D}_r] = 0, \qquad \hat{C}, \hat{D} \in \{\hat{n}, \hat{a}, \hat{a}^{+}\}. \tag{8}$$

At the same time the operators of the different modes commute with each other: $[\hat{C}_i, \hat{D}_j] = 0\ (i \neq j)$. We must note that the 'displaced' operators $\hat{A}_i$ $\left(\hat{A}_i^{+}\right)$ and $\hat{N}_i = \hat{A}_i^{+}\hat{A}_i$ obey the same commutation relations as $\hat{a}_i$ $\left(\hat{a}_i^{+}\right)$ and $\hat{n}$ as shown in equations (6) and (7). The operators in $hw(4)_l$ and $hw(4)_r$ in terms of $\hat{A}_i$ $\left(\hat{A}_i^{+}\right)$ and $\hat{N}_i = \hat{A}_i^{+}\hat{A}_i$ can be composed into a direct sum of a four-simple-Lie algebra $SU(1,1)$ and a two-Lie algebra $SU(2)$:

$$\begin{aligned}
&\hat{K}_1^{-} = \hat{A}_{1l}\hat{A}_{1r}^{+}, && \hat{K}_1^{+} = \hat{A}_{1l}^{+}\hat{A}_{1r}, && \hat{K}_1^{0} = \tfrac{1}{2}(\hat{N}_{1l} + \hat{N}_{1r}),\\
&\hat{K}_2^{-} = \hat{A}_{2l}\hat{A}_{2r}^{+}, && \hat{K}_2^{+} = \hat{A}_{2l}^{+}\hat{A}_{2r}, && \hat{K}_2^{0} = \tfrac{1}{2}(\hat{N}_{2l} + \hat{N}_{2r}),\\
&\hat{K}_3^{-} = \hat{A}_{1l}\hat{A}_{2r}^{+}, && \hat{K}_3^{+} = \hat{A}_{1l}^{+}\hat{A}_{2r}, && \hat{K}_3^{0} = \tfrac{1}{2}(\hat{N}_{1l} + \hat{N}_{2r}),\\
&\hat{K}_4^{-} = \hat{A}_{2l}\hat{A}_{1r}^{+}, && \hat{K}_4^{+} = \hat{A}_{2l}^{+}\hat{A}_{1r}, && \hat{K}_4^{0} = \tfrac{1}{2}(\hat{N}_{2l} + \hat{N}_{1r}),\\
&\hat{J}_l^{-} = \hat{A}_{1l}\hat{A}_{2l}^{+}, && \hat{J}_l^{+} = \hat{A}_{1l}^{+}\hat{A}_{2l}, && \hat{J}_l^{0} = \tfrac{1}{2}(\hat{N}_{1l} - \hat{N}_{2l}),\\
&\hat{J}_r^{-} = \hat{A}_{1r}\hat{A}_{2r}^{+}, && \hat{J}_r^{+} = \hat{A}_{1r}^{+}\hat{A}_{2r}, && \hat{J}_r^{0} = -\tfrac{1}{2}(\hat{N}_{1r} - \hat{N}_{2r}).
\end{aligned} \tag{9}$$

They satisfy the relations

$$\left[\hat{K}_i^{0}, \hat{K}_i^{\pm}\right] = \pm\hat{K}_i^{\pm}, \qquad \left[\hat{K}_i^{-}, \hat{K}_i^{+}\right] = 2\hat{K}_i^{0}, \qquad (i = 1, 2, 3, 4) \tag{10}$$

$$\left[\hat{J}_j^{0}, \hat{J}_j^{\pm}\right] = \pm\hat{J}_j^{\pm}, \qquad \left[\hat{J}_j^{+}, \hat{J}_j^{-}\right] = 2\hat{J}_j^{0}, \qquad (j = l, r). \tag{11}$$

Then equation (5) can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \left[\sum_{i=1}^{2}\gamma_i\left(2\hat{A}_{il}\hat{A}_{ir}^{+} - \hat{A}_{il}^{+}\hat{A}_{il} - \hat{A}_{ir}^{+}\hat{A}_{ir}\right)\right.\\
&\quad \left. + \sum_{i,j=1(i\neq j)}^{2}\left(2\gamma_3\hat{A}_{il}\hat{A}_{jr}^{+} - g\hat{A}_{il}^{+}\hat{A}_{jl} - g^{*}\hat{A}_{ir}^{+}\hat{A}_{jr}\right)\right]\hat{\rho}\\
&= \hat{\Gamma}\hat{\rho}
\end{aligned} \tag{12}$$

where the Liouvillian superoperator $\hat{\Gamma}$ is

$$\begin{aligned}
\Gamma = 2\Big[&\gamma_1\hat{K}_1^{-} + \gamma_2\hat{K}_2^{-} + \gamma_3\hat{K}_3^{-} + \gamma_3\hat{K}_4^{-} - \frac{g}{2}\left(\hat{J}_l^{-} + \hat{J}_l^{+}\right) - \frac{g^{*}}{2}\left(\hat{J}_r^{-} + \hat{J}_r^{+}\right)\\
&- \gamma_1\hat{K}_1^{0} - \gamma_2\hat{K}_2^{0} + \frac{\gamma_1 + \gamma_2}{2}\Big]
\end{aligned} \tag{13}$$

4. Adiabatic solution to master equation

In the following we use similarity transformations to diagonalize the Liouvillian superoperator $\hat{\Gamma}$ in equation (13) and to solve the instantaneous eigenvectors problem. We use the operator

$$\hat{U}(t) = \hat{U}_1(t)\hat{U}_2(t)\hat{U}_3(t)\hat{U}_4(t) \tag{14}$$

where

$$\hat{U}_1(t) = \exp\left(\alpha_+\hat{J}_l^+\right)\exp\left(\alpha_-\hat{J}_l^-\right), \tag{15}$$

$$\hat{U}_2(t) = \exp\left(\beta_+\hat{J}_r^+\right)\exp\left(\beta_-\hat{J}_r^-\right), \tag{16}$$

$$\hat{U}_3(t) = \exp\left(\theta_2\hat{K}_4^-\right)\exp\left(\theta_1\hat{K}_3^-\right), \tag{17}$$

$$\hat{U}_4(t) = \exp\left(\varphi_2\hat{K}_2^-\right)\exp\left(\varphi_1\hat{K}_1^-\right) \tag{18}$$

to carry out the similarity transformation

$$\hat{\Gamma}' = \hat{U}(t)^{-1}\hat{\Gamma}\hat{U}(t), \qquad \hat{\rho}' = \hat{U}(t)^{-1}\hat{\rho}. \tag{19}$$

When the parameters $\alpha_\pm$, $\beta_\pm$, $\theta_{1,2}$ and $\varphi_{1,2}$ satisfy

$$\alpha_-(\gamma_1-\gamma_2)(1+\alpha_+\alpha_-) - g(1+\alpha_+\alpha_-)^2 + g\alpha_-^2 = 0, \tag{20}$$

$$\alpha_+(\gamma_2-\gamma_1) + g\left(\alpha_+^2-1\right) = 0, \tag{21}$$

$$\beta_-(\gamma_2-\gamma_1)(1+\beta_+\beta_-) - g^*(1+\beta_+\beta_-)^2 + g^*\beta_-^2 = 0, \tag{22}$$

$$\beta_+(\gamma_1-\gamma_2) + g^*\left(\beta_+^2-1\right) = 0, \tag{23}$$

$$\begin{aligned}\gamma_1\beta_-(1+\alpha_+\alpha_-) + \gamma_3\alpha_-\beta_- + [\gamma_2\alpha_- + \gamma_3(1+\alpha_+\alpha_-)](1+\beta_+\beta_-)\\ + \frac{\theta_1}{2}(\gamma_1+\gamma_2-g^*\beta_+-g\alpha_+) = 0,\end{aligned} \tag{24}$$

$$\gamma_2\beta_+ + \gamma_3\alpha_+\beta_+ + \gamma_1\alpha_+ + \gamma_3 + \frac{\theta_2}{2}(\gamma_1+\gamma_2+g^*\beta_++g\alpha_+) = 0, \tag{25}$$

$$(\gamma_1+\gamma_3\beta_+)(1+\alpha_+\alpha_-) + \gamma_3\alpha_- + \gamma_2\alpha_-\beta_+ + \frac{\varphi_1}{2}(2\gamma_1+g^*\beta_+-g\alpha_+) = 0, \tag{26}$$

$$(\gamma_2+\gamma_3\alpha_+)(1+\beta_+\beta_-) + (\gamma_1\alpha_++\gamma_3)\beta_- + \frac{\varphi_2}{2}(2\gamma_2-g^*\beta_++g\alpha_+) = 0, \tag{27}$$

the transformed superoperator is diagonalized in the 'displaced' number representation generated by the eigenbasis of $\hat{N}_{il}$ $(\hat{N}_{ir})$ $(i=1,2)$:

$$\hat{\Gamma}' = (g\alpha_+-\gamma_1)\hat{N}_{1l} - (\gamma_1+g^*\beta_+)\hat{N}_{1r} - (\gamma_2+g\alpha_+)\hat{N}_{2l} + (g^*\beta_+-\gamma_2)\hat{N}_{2r} + \gamma_1+\gamma_2. \tag{28}$$

To obtain the eigensolutions of the Liouvillian superoperator $\hat{\Gamma}$ in equation (13), we first solve the eigenvalue problem of the transformed superoperator $\hat{\Gamma}'$ in equation (28). The eigenvectors of the 'displaced' number operators $\hat{N}_{il}$ $(\hat{N}_{ir})$ $(i=1,2)$ in equation (28) can be directly calculated as follows. From the relations

$$\hat{N}_{il} = \hat{D}_{il}\hat{n}_{il}\hat{D}_{il}^{-1}, \tag{29}$$

$$\hat{N}_{ir} = \hat{D}_{ir}\hat{n}_{ir}\hat{D}_{ir}^{-1} \tag{30}$$

where

$$\hat{D}_{1l} = \exp\left(-x\hat{a}_{1l}^+\right)\exp(x^*\hat{a}_{1l}) \qquad \hat{D}_{2l} = \exp\left(-y\hat{a}_{2l}^+\right)\exp(y^*\hat{a}_{2l}) \tag{31}$$

and

$$\hat{D}_{1r} = \exp\left(x\hat{a}_{1r}^{+}\right)\exp(-x^{*}\hat{a}_{1r}) \qquad \hat{D}_{2r} = \exp\left(y\hat{a}_{2r}^{+}\right)\exp(-y^{*}\hat{a}_{2r}), \tag{32}$$

we can obtain the eigenvectors of the 'displaced' number operators $\hat{N}_{il}$ ($\hat{N}_{ir}$) ($i = 1, 2$), i.e. the 'displaced' number representation bases:

$$|N_{1l}\rangle = \exp\left(-x\hat{a}_{1l}^{+}\right)\exp(x^{*}\hat{a}_{1l})|n_{1l}\rangle \qquad |N_{2l}\rangle = \exp\left(-y\hat{a}_{2l}^{+}\right)\exp(y^{*}\hat{a}_{2l})|n_{2l}\rangle \tag{33}$$

$$\langle N_{1r}| = \exp\left(x\hat{a}_{1r}^{+}\right)\exp(-x^{*}\hat{a}_{1r})\langle n_{1r}| \qquad \langle N_{2r}| = \exp\left(y\hat{a}_{2r}^{+}\right)\exp(-y^{*}\hat{a}_{2r})\langle n_{2r}| \tag{34}$$

where $|n_{il}\rangle$ ($\langle n_{ir}|$) are eigenvectors of the number operator $\hat{n}_l$ ($\hat{n}_r$). The actions of the operators $\hat{A}_{il}$, $\hat{A}_{il}^{+}$ $\left(\hat{A}_{ir}, \hat{A}_{ir}^{+}\right)$ and $\hat{N}_{il}$ ($\hat{N}_{ir}$) ($i = 1, 2$) on these states are

$$\hat{N}_{il}|N_{il}\rangle = n_{il}|N_{il}\rangle, \qquad \hat{A}_{il}|N_{il}\rangle = \sqrt{n_{il}}|N_{il} - 1\rangle, \qquad \hat{A}_{il}^{+}|N_{il}\rangle = \sqrt{n_{il} + 1}|N_{il} + 1\rangle, \tag{35}$$

and

$$\hat{N}_{ir}\langle N_{ir}| = (n_{ir} + 1)\langle N_{ir}|, \qquad \hat{A}_{ir}\langle N_{ir}| = \sqrt{n_{ir} + 1}\langle N_{ir} + 1|, \qquad \hat{A}_{ir}^{+}\langle N_{ir}| = \sqrt{n_{ir}}\langle N_{ir} - 1|, \tag{36}$$

where

$$|N_{1l} \pm 1\rangle = \exp\left(-x\hat{a}_{1l}^{+}\right)\exp(x^{*}\hat{a}_{1l})|n_{1l} \pm 1\rangle$$
$$|N_{2l} \pm 1\rangle = \exp\left(-y\hat{a}_{2l}^{+}\right)\exp(y^{*}\hat{a}_{2l})|n_{2l} \pm 1\rangle$$

and

$$\langle N_{1r} \pm 1| = \exp\left(x\hat{a}_{1r}^{+}\right)\exp(-x^{*}\hat{a}_{1r})\langle n_{1r} \pm 1|$$
$$\langle N_{2r} \pm 1| = \exp\left(y\hat{a}_{2r}^{+}\right)\exp(-y^{*}\hat{a}_{2r})\langle n_{2r} \pm 1|.$$

Then the eigenvectors of the transformed superoperator in equation (28) are the following supervectors in Liouvillian space,

$$\hat{P}_{n_{1l},n_{1r},n_{2l},n_{2r}} = |N_{1l}\rangle\langle N_{1r}| \otimes |N_{2l}\rangle\langle N_{2r}|, \qquad \text{i.e.} \quad \hat{\Gamma}'\hat{P}_{n_{1l},n_{1r},n_{2l},n_{2r}} = \lambda_{n_{1l},n_{1r},n_{2l},n_{2r}}\hat{P}_{n_{1l},n_{1r},n_{2l},n_{2r}} \tag{37}$$

where

$$\begin{aligned}\lambda_{n_{1l},n_{1r},n_{2l},n_{2r}} &= (g\alpha_{+} - \gamma_1)n_{1l} - (\gamma_1 + g^{*}\beta_{+})(n_{1r} + 1) - (\gamma_2 + g\alpha_{+})n_{2l}\\ &\quad + (g^{*}\beta_{+} - \gamma_2)(n_{2r} + 1) + \gamma_1 + \gamma_2\\ &= -(\gamma_1 - g\alpha_{+})n_{1l} - (\gamma_1 + g^{*}\beta_{+})n_{1r} - (\gamma_2 + g\alpha_{+})n_{2l} - (\gamma_2 - g^{*}\beta_{+})n_{2r}.\end{aligned} \tag{38}$$

The solutions of equations (40) and (41) show that the real parts of the complex numbers in the four brackets ( ) of equation (38) are all positive, which guarantees the existence of the equilibrium state of the master equation. From equations (14)–(19), (35)–(37), we can obtain the instantaneous eigenvectors of the Liouvillian superoperator $\hat{\Gamma}$ in equation (13) as follows:

$$\hat{\chi}_m = \hat{U}_1(t)\hat{U}_2(t)\hat{U}_3(t)\hat{U}_4(t)\hat{P}_{n_{1l},n_{1r},n_{2l},n_{2r}} \tag{39}$$

with the eigenvalues $\lambda_{n_{1l},n_{1r},n_{2l},n_{2r}}$ given in equation (38).

Equations (20)–(23) have two solutions:

$$\alpha_{+} = \frac{\Delta\gamma - \delta}{2g}, \quad \alpha_{-} = \frac{g}{\delta} \qquad \text{or} \qquad \alpha_{+} = \frac{\Delta\gamma + \delta}{2g}, \quad \alpha_{-} = -\frac{g}{\delta}, \tag{40}$$

$$\beta_{+} = -\frac{\Delta\gamma + \delta^{*}}{2g^{*}}, \quad \beta_{-} = \frac{g^{*}}{\delta^{*}} \qquad \text{or} \qquad \beta_{+} = \frac{\delta^{*} - \Delta\gamma}{2g^{*}}, \quad \beta_{-} = -\frac{g^{*}}{\delta^{*}}, \tag{41}$$

with $\Delta\gamma = \gamma_1 - \gamma_2$ and $\delta = \sqrt{4g^2 + \Delta\gamma^2}$. The solutions of equations (24)–(27) are given by

$$\theta_1 = -2\{\gamma_1\beta_-(1+\alpha_+\alpha_-) + \gamma_3\alpha_-\beta_- + [\gamma_2\alpha_- + \gamma_3(1+\alpha_+\alpha_-)] \\ \times (1+\beta_+\beta_-)\}/(\gamma_1+\gamma_2 - g^*\beta_+ - g\alpha_+), \tag{42}$$

$$\theta_2 = -2(\gamma_2\beta_+ + \gamma_3\alpha_+\beta_+ + \gamma_1\alpha_+ + \gamma_3)/(\gamma_1+\gamma_2+g^*\beta_+ + g\alpha_+), \tag{43}$$

$$\varphi_1 = -2[(\gamma_1+\gamma_3\beta_+)(1+\alpha_+\alpha_-) + \gamma_3\alpha_- + \gamma_2\alpha_-\beta_+]/(2\gamma_1 + g^*\beta_+ - g\alpha_+), \tag{44}$$

$$\varphi_2 = -2[(\gamma_2+\gamma_3\alpha_+)(1+\beta_+\beta_-) + (\gamma_1\alpha_+ + \gamma_3)\beta_-]/(2\gamma_2 - g^*\beta_+ + g\alpha_+). \tag{45}$$

The result that two similarity transformations exist to diagonalize the same Liouvillian superoperator $\hat{\Gamma}$ and to yield two sets of eigenvalues indicates that the superoperator $\hat{\Gamma}$ can be diagonalized not only by 'rotating to one chosen direction' but also by 'rotating to its opposite direction' in the group generator space since the main part of the superoperator $\hat{\Gamma}$ is a vector in this space.

5. Geometrical phase for dissipative two-mode optical system

Here we consider a slowly varying parameter space $\vec{R}(t) = (F(t), f_{1,2}(t), f_{1,2}(t)^*)$. The adiabatic solution to equation (4) or (12) is [12]

$$\hat{\xi}_{n_{1l},n_{1r},n_{2l},n_{2r}} = \exp\left[\int_0^\tau \lambda_{n_{1l},n_{1r},n_{2l},n_{2r}}(t')\,\mathrm{d}t'\right]\exp\left[\mathrm{i}\phi_{n_{1l},n_{1r},n_{2l},n_{2r}}(\tau)\right]\hat{\chi}_{n_{1l},n_{1r},n_{2l},n_{2r}} \tag{46}$$

where the adiabatic geometrical phase $\phi_{n_{1l},n_{1r},n_{2l},n_{2r}}$ is [12]

$$\begin{aligned}\phi_{n_{1l},n_{1r},n_{2l},n_{2r}} &= \frac{\mathrm{i}}{2}\int_{\vec{R}(0)}^{\vec{R}(\tau)} \mathrm{Tr}\left[\hat{\chi}^+_{n_{1l},n_{1r},n_{2l},n_{2r}}(t)\nabla_{\vec{R}}\hat{\chi}_{n_{1l},n_{1r},n_{2l},n_{2r}}(t)\right]\mathrm{d}\vec{R} \\ &= \frac{\mathrm{i}}{2}\left[\int_{F(0)}^{F(\tau)} \mathrm{Tr}\left[\hat{\chi}^+_{n_{1l},n_{1r},n_{2l},n_{2r}}\frac{\partial}{\partial F}\hat{\chi}_{n_{1l},n_{1r},n_{2l},n_{2r}}\right]\mathrm{d}F\right. \\ &\quad + \sum_{i=1}^{2}\left(\int_{f_i(0)}^{f_i(\tau)} \mathrm{Tr}\left[\hat{\chi}^+_{n_{1l},n_{1r},n_{2l},n_{2r}}\frac{\partial}{\partial f_i}\hat{\chi}_{n_{1l},n_{1r},n_{2l},n_{2r}}\right]\mathrm{d}f_i\right. \\ &\quad \left.\left. + \int_{f_i(0)^*}^{f_i(\tau)^*} \mathrm{Tr}\left[\hat{\chi}^+_{n_{1l},n_{1r},n_{2l},n_{2r}}\frac{\partial}{\partial f_i^*}\hat{\chi}_{n_{1l},n_{1r},n_{2l},n_{2r}}\right]\mathrm{d}f_i^*\right)\right] \\ &= \frac{\mathrm{i}}{2}(n_{1l}-n_{2l})\int_{F(0)}^{F(\tau)}\alpha_-\dot{\alpha}_+\,\mathrm{d}F + \frac{\mathrm{i}}{2}(n_{2r}-n_{1r})\int_{F(0)}^{F(\tau)}\beta_-\dot{\beta}_+\,\mathrm{d}F\end{aligned} \tag{47}$$

where the derivative is taken with respect to the parameter F. The external driving fields described by $f_i(t)$ $f_i^*(t)$ do not contribute to the geometrical phase due to the fact that their algebras $hw(4)_l$ and $hw(4)_r$ are anti-isomorphic to each other. If we assume that the system initially evolves from a single two-mode Fock state, i.e. $|m\rangle\langle m| \otimes |n\rangle\langle n|$, the geometrical phase in equation (47) becomes

$$\phi_{m,n} = \frac{\mathrm{i}}{2}(m-n)\int_{F(0)}^{F(\tau)}(\alpha_-\dot{\alpha}_+ - \beta_-\dot{\beta}_+)\,\mathrm{d}F. \tag{48}$$

From equations (40), (41) we get two sets of solutions of $\alpha_-\dot{\alpha}_+$ and $\beta_-\dot{\beta}_+$:

$$(1) \quad \alpha_-\dot{\alpha}_+ = \frac{\mathrm{i}\Delta\gamma(\Delta\gamma-\delta)}{2g\delta^2}, \qquad \beta_-\dot{\beta}_+ = -\frac{\mathrm{i}\Delta\gamma(\Delta\gamma-\delta^*)}{2g^*\delta^{*2}}; \tag{49}$$

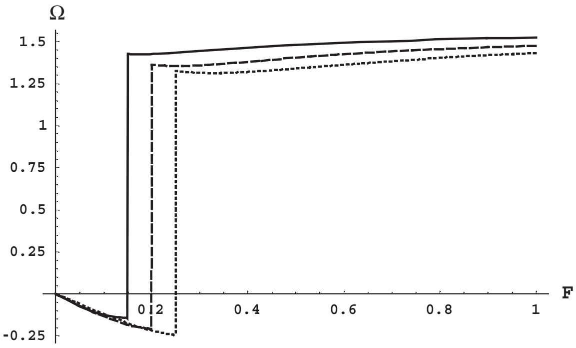

Figure 1. Ω factor of geometrical phase versus coupling constant F with $\gamma_2 = 0.1$, $\gamma_3 = \sqrt{\gamma_1\gamma_2}$ and with different γ_1: $\gamma_1 = 0.2$ (solid curve); $\gamma_1 = 0.3$ (dashed curve); $\gamma_1 = 0.4$ (dotted curve).

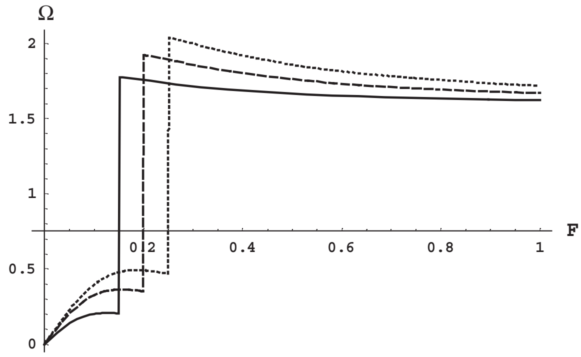

Figure 2. Ω factor of geometrical phase versus coupling constant F with $\gamma_1 = 0.1$, $\gamma_3 = \sqrt{\gamma_1\gamma_2}$ and different γ_2: $\gamma_2 = 0.2$ (solid curve); $\gamma_2 = 0.3$ (dashed curve); $\gamma_2 = 0.4$ (dotted curve).

$$(2)\quad \alpha_-\dot{\alpha}_+ = \frac{\mathrm{i}\Delta\gamma(\Delta\gamma+\delta)}{2g\delta^2}, \qquad \beta_-\dot{\beta}_+ = -\frac{\mathrm{i}\Delta\gamma(\Delta\gamma+\delta^*)}{2g^*\delta^{*2}}. \tag{50}$$

It can be seen that the phase is related to the photon number difference of the two modes, their coupling constant as well as their damping constants.

To deeply investigate the dependence of the geometrical phase on the coupling constant, we plot the factor of geometrical phase in equation (48), the integral $\Omega = \int_{F(0)}^{F(\tau)} (\alpha_-\dot{\alpha}_+ - \beta_-\dot{\beta}_+)\,\mathrm{d}F$ versus F, in figure 1 with $\gamma_2 = 0.1$ and the relation $\gamma_3 = \sqrt{\gamma_1\gamma_2}$. From the solid curve with $\gamma_1 = 0.2$ in figure 1, we see that Ω decreases monotonically from 0 with F in the region $0 < F < 0.15$, but increases drastically at the point $F = 0.15$. From $F = 0.15$ to $F = 1$, the solid curve increases slowly with F. The cases with $\gamma_1 = 0.3$ and $\gamma_1 = 0.4$ are displayed by the dashed and dotted curves in figure 1, respectively. They are similar to the case with $\gamma_1 = 0.2$, but they increase drastically at the points $F = 0.2$ and $F = 0.25$, respectively.

In figure 2, γ_2 assumes different values and γ_1 is set to 0.1. The solid, dash and dotted curves correspond to $\gamma_2 = 0.2$, 0.3 and 0.4, respectively. This time the curves increase slowly as F varies from 0 to 0.15, 0.2 and 0.25 respectively from where the curves increase drastically. Afterwards, they decrease slowly with increasing F. The comparison of figure 1 with figure 2 shows that if the values of γ_1 in figure 1 are set equal to those of

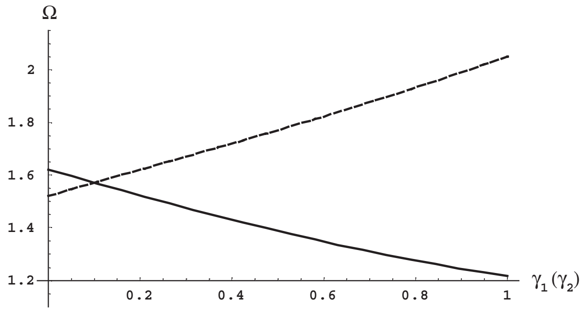

Figure 3. Ω factor of geometrical phase versus damping constant γ_1 with $\gamma_2 = 0.1$, $\gamma_3 = \sqrt{\gamma_1\gamma_2}$ and $k = 0.1$ (solid curve); and Ω versus damping constant γ_2 with $\gamma_1 = 0.1$, $\gamma_3 = \sqrt{\gamma_1\gamma_2}$ and $k = 0.1$ (dashed curve).

γ_2 in figure 2, the corresponding curves increase drastically at the same points of F. As we extend the range of F to 10 (not plotted here), we find that all the curves in figures 1 and 2 asymptotically approach about 1.57, which implies that the geometrical phase for a given two-mode photon state becomes constant as the value of F increases beyond a certain value.

The dependence of Ω on damping constants γ_1 and γ_2 is displayed in figure 3, which indicates that Ω decreases with γ_1 but increases with γ_2 in the same interval [0, 1], its variation is opposite to the contributions from the initial photon numbers to the geometrical phase.

The dependence of the geometrical phase on the photon number difference $m - n$ of the two modes can be used to judge whether the photon numbers of both modes are equal just by identifying the null geometrical phase for nonzero Ω. It can also be determined which mode has more photons than the other, just from the positive or negative sign of the phase. The dependence of geometrical phase on photon number difference $m - n$ of the two modes is mainly due to the $SU(2)$ algebraic structure of the two-mode coupling term in equation (1). This term can also be understood as polarization transfer of the two modes with different polarization of the same spatial mode [26]. Therefore, the geometric phase can be produced from the polarization rotation of the field propagating slowly along an anisotropic dielectric medium. From equation (48), we anticipate that the geometrical phase can be measured by an interference experiment between a two-mode state evolving from $|m\rangle\langle m| \otimes |n\rangle\langle n|$ with unequal numbers of photons ($m \neq n$) and a two-mode state evolving from $|m\rangle\langle m| \otimes |m\rangle\langle m|$ with equal numbers of photons which give null geometrical phase. It should be noted that the above information is realistic in the sense that the effect of the dissipative background is included.

6. Summary and conclusion

By using the algebraic dynamical method we have found the dynamical algebraic structure of the dissipative two-mode optical system and converted its master equation into a Schrödinger-like equation. Then we have obtained the exact eigenvectors of the Liouvillian superoperator and investigated their geometrical phases. The geometric phase is related to the photon number difference of the two modes, their damping constants and their coupling constant. In view of that the two-mode cavity system can be used as a 2 q-bit unit in quantum computation, and the quantum control gate made from the geometric phase is robust against dynamical disturbance of the environment. The study of the information about the dissipative two-mode optical cavity system itself, especially its geometric phase, is of great interest and significance in the fields of quantum information and quantum computation. In the forthcoming papers we shall use this model to investigate the free-interaction measurement, reversible decoherence of Schrödinger cat states, and quantum information transfer and quantum gate under the effects of the dissipative background and the external coherent driving fields.

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